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Vibrations of Mindlin rectangular plates with elastically restrained edges using static Timoshenko beam functions with the Rayleigh–Ritz method

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Abstract

A set of static Timoshenko beam functions is developed as the admissible functions for the free vibration analysis of Mindlin rectangular plates with uniform elastic (translational and/or rotational) edge constraints by the use of the Rayleigh–Ritz method. This set of beam functions is made up of two parts: the transverse deflection functions and rotation functions, which are the static solutions of a Timoshenko beam under the transverse loads of a Fourier sinusoidal series distributed along the length of the beam, therefore the effect of shear deformation on the admissible functions is taken into account. In fact, the beam is considered to be a unit width strip taken from the rectangular plate in a direction parallel to the edge of the plate and has the end supports corresponding to the edge conditions of the particular plate under consideration. It can be seen that the method is sound theoretically. Each of the beam functions is a polynomial function of not more than third order plus a sine or cosine function for all cases and the change of the boundary supports only results in a corresponding change of the coefficients of the low-order polynomial. Therefore, a unified program with little computational efforts can be easily developed for the vibration analysis of Mindlin rectangular plates with arbitrary boundary conditions and thickness-span ratios. The accuracy and efficiency of the proposed method are demonstrated by comparison and convergency studies. Finally, sets of reasonably accurate natural frequencies are presented, which can serve as a supplement to the database of vibration characteristics of moderately thick plates. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Mindlin plate; Rectangular plate; Free vibration; Eigenfrequencies; Elastically restrained edges; Static Timoshenko beam functions; Rayleigh–Ritz method

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1. Introduction

Rectangular plates have been extensively used in many branches of engineering such as civil, mechanical, aeronautical and marine engineering. The vibration analysis is essential for avoiding damages to the structures under external dynamic loads.

It is well known that the classical plate theory (Timoshenko and Woinowsky-Kroeger, 1959) has been successfully applied to the thin plate analysis, in which the assumption of a straight line originally normal to the median surface of plate remains straight and normal after deformation is adopted, which means that all transverse shear strains are equal to zero in the plate. However, with the increase of the thickness-span ratio of the plate, shear deformation becomes increasingly important. In such a case, the classical plate theory inevitably results in some errors. Moreover, the shear effect should be considered when greater accuracy of modes and/or dynamic responses under high-frequency excitations is required, even for the thin plates. Relaxing the normality assumption, Mindlin (1951), Mindlin et al. (1956) presented a so-called first order shear deformation theory for moderately thick plates, in which the rotary inertia effect was also considered. The original normal line has now become a straight non-normal line which is a close approximation to the actual curved-line distribution of the shear deformation along the thickness of the plate in an average sense. A shear correction factor is then introduced to compensate for the errors resulting from the approximation made on the non-uniform shear strain distribution. The Mindlin plate theory has been widely employed to analyze the static and dynamic properties of moderately thick plates because of the fact that this theory not only retains mathematically the two-dimensionality but also has sufficient accuracy for the plates with moderate thickness.

Substantial references about vibrations of rectangular thin plates (Leissa, 1973) can be found. Some investigations on the vibration of rectangular thin plates with elastically restrained edges by the use of the Rayleigh–Ritz method have also been made (Warburton and Edney, 1984; Laurra and Gross, 1981; Zhou, 1995). Gorman (1989, 1990) studied the vibration of rectangular thin plates with symmetrically and asymmetrically distributed uniform elastic edge constraints by using the superposition method. In comparison, available work on vibrations of Mindlin rectangular plates is rather limited (Liew et. al., 1995), especially for Mindlin rectangular plates with elastically restrained edges. Magrab (1977) studied the simply supported Mindlin orthotropic rectangular plates with rotational constraints by using the Galerkin technique. Recently, Saha et al. (1996) used the vibrating Timoshenko beam functions as the admissible functions to study the free vibration of Mindlin rectangular plates with uniformly distributed elastic restraints along the edges and Chung et al. (1993) used the recurrence polynomials as the admissible functions to investigate the vibration of orthotropic Mindlin rectangular plates with elastic rotational restraints on edges by the Rayleigh–Ritz method, respectively. Moreover, Xiang et al. (1997) used the two-dimensional polynomials as the admissible functions to study the vibrations of rectangular Mindlin plates with elastic edge supports by the Ritz method and the same problems were investigated by Gorman (1997a,b) using the superposition method.

In this paper, a set of static Timoshenko beam functions is developed as the admissible functions for the vibration analysis of Mindlin rectangular plates with elastically restrained edges by using the Rayleigh–Ritz method. Each of the beam functions is only a polynomial function of not more than third order plus a sine or a cosine function, in which the coefficients of the polynomial are uniquely determined by the boundary conditions of the plate. Both simple mathematical representation and rapid convergency demonstrate the small computational efforts required. Comparison of results with those available from literature shows the high accuracy of the present approach.

2. The Rayleigh–Ritz formulae for Mindlin rectangular plates

Consider an isotropic moderately thick rectangular plate with uniform thickness h, length a and width b, which lies in the x–y plane, as shown in Fig. 1. The surfaces of the plate are the planes $z = \pm h/2$. The edges of the plate are elastically restrained by both translational springs and rotational springs. It is obvious that the classical boundary conditions of the plate represent only the limiting cases as the elastic restraints approach their natural limits of zero or infinity. According to the Mindlin plate theory, three fundamental quantities w, ψ_x and ψ_y are utilized, which describe the transverse displacement of the plate median surface and the rotations of the cross-section, respectively, along the x direction and the y direction. For free vibration of the plate, the solutions of w, ψ_x and ψ_y have the form of

$$w(x,y,t) = W(x,y)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \psi_x(x,y,t) = \Psi_x(x,y)\mathrm{e}^{\mathrm{i}\omega t}/a, \qquad \psi_y(x,y,t) = \Psi_y(x,y)\mathrm{e}^{\mathrm{i}\omega t}/b, \tag{1}$$

where $W(x,y)$, $\Psi_x(x,y)$ and $\Psi_y(x,y)$ are the dynamic displacement function of the plate and the dynamic rotation functions, respectively, along the x direction and the y direction. ω is the radian natural frequency of the plate and $\mathrm{i} = \sqrt{-1}$.

The energy functional Π for a Mindlin plate with elastically restrained edges can be written in terms of the maximum strain energy $U_{\max}$ and the maximum kinetic energy $T_{\max}$ as

$$\Pi = U_{\max} - T_{\max}. \tag{2}$$

Introduce the non-dimensional coordinates $\xi = x/a$ and $\eta = y/b$, $U_{\max}$ and $T_{\max}$ are given, respectively, by

$$\begin{aligned}
U_{\max} = {} & \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 \left\{ \left(\frac{\partial \Psi_x}{\partial \xi} + \left(\frac{a}{b}\right)^2 \frac{\partial \Psi_y}{\partial \eta} \right)^2 - 2(1-\nu)\left(\frac{a}{b}\right)^2 \left[\frac{\partial \Psi_x}{\partial \xi}\frac{\partial \Psi_y}{\partial \eta} - \frac{1}{4}\left(\frac{\partial \Psi_x}{\partial \eta} + \frac{\partial \Psi_y}{\partial \xi} \right)^2 \right] \right. \\
& \left. + \frac{\kappa G h}{D} a^2 \left[\left(\Psi_x + \frac{\partial W}{\partial \xi} \right)^2 + \left(\frac{a}{b}\right)^2 \left(\Psi_y + \frac{\partial W}{\partial \eta} \right)^2 \right] \right\} \mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{2}k_{x0}^{\mathrm{t}} b \int_0^1 W^2\big|_{\xi=0}\,\mathrm{d}\eta \\
& + \frac{1}{2}k_{x1}^{\mathrm{t}} b \int_0^1 W^2\big|_{\xi=1}\,\mathrm{d}\eta + \frac{1}{2}k_{y0}^{\mathrm{t}} a \int_0^1 W^2\big|_{\eta=0}\,\mathrm{d}\xi + \frac{1}{2}k_{y1}^{\mathrm{t}} a \int_0^1 W^2\big|_{\eta=1}\,\mathrm{d}\xi + \frac{1}{2}\frac{k_{x0}^{\mathrm{r}} b}{a^2}\int_0^1 \Psi_x^2\big|_{\xi=0}\,\mathrm{d}\eta \\
& + \frac{1}{2}\frac{k_{x1}^{\mathrm{r}} b}{a^2}\int_0^1 \Psi_x^2\big|_{\xi=1}\,\mathrm{d}\eta + \frac{1}{2}\frac{k_{y0}^{\mathrm{r}} a}{b^2}\int_0^1 \Psi_y^2\big|_{\eta=0}\,\mathrm{d}\xi + \frac{1}{2}\frac{k_{y1}^{\mathrm{r}} a}{b^2}\int_0^1 \Psi_y^2\big|_{\eta=1}\,\mathrm{d}\xi, \\
T_{\max} = {} & \frac{1}{2}\rho h \omega^2 ab \int_0^1\int_0^1 \left[W^2 + \frac{1}{12}\left(\frac{h}{b}\right)^2 \left(\Psi_x^2/\lambda^2 + \Psi_y^2 \right) \right] \mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned} \tag{3}$$

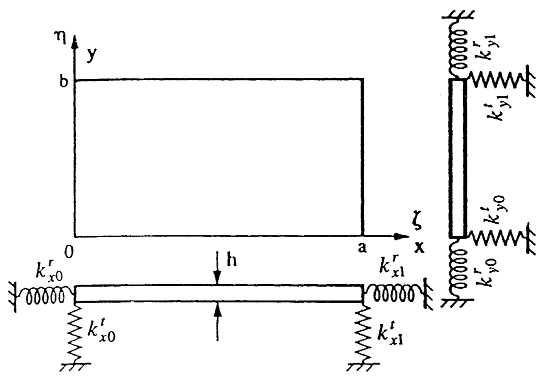

Fig. 1. The sketch of moderately thick rectangular plate with elastically restrained edges.

in which, $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate, ν is Poisson's ratio, E is Young's modulus and $G = E/[2(1+\nu)]$ is the elastic modulus of shear. κ is the shear correction factor and ρ is the density of the plate per unit volume, $k^{\rm t}_{x0}$, $k^{\rm t}_{x1}$ are the translational stiffness, respectively, along the edge, $x = 0$ and the edge, $x = a$ and $k^{\rm t}_{y0}$, $k^{\rm t}_{y1}$ are those, respectively, along the edge, $y = 0$ and the edge, $y = b$. $k^{\rm r}_{x0}$, $k^{\rm r}_{x1}$ are the rotational stiffness, respectively, along the edge, $x = 0$ and the edge, $x = a$ and $k^{\rm r}_{y0}$, $k^{\rm r}_{y1}$ are those, respectively, along the edge, $y = 0$ and the edge, $y = b$.

Assume that functions W, Ψ_x and Ψ_y are all separable in variables and can take the following forms, respectively

$$W(\xi,\eta) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}X_i(\xi)Y_j(\eta), \qquad \Psi_x(\xi,\eta) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} d_{ij}\Phi_i(\xi)Y_j(\eta)/a,$$

$$\Psi_y(\xi,\eta) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e_{ij}X_i(\xi)\Psi_j(\eta)/b, \tag{4}$$

where c_{ij}, d_{ij} and e_{ij} $(i, j = 1, 2, 3, \ldots)$ are the unknown constants, $X_i(\xi)$, $\Phi_i(\xi)$ and $Y_j(\eta)$, $\Psi_j(\eta)$ are the appropriate admissible functions, respectively, in the ξ and η directions. Substituting Eq. (4) into Eqs. (2) and (3) and truncating the number of terms of the series in x and y directions up to $I+1$ and $J+1$ respectively, then setting the first variation of the energy functional Π zero value results in the following eigenvalue equation

$$\left([K] - \Omega^2(\lambda\pi)^4[M]\right)\begin{Bmatrix}\{c\}\\ \{d\}\\ \{e\}\end{Bmatrix} = \{0\}, \tag{5}$$

in which, $[K]$ is the stiffness matrix of the plate, $[M]$ is the mass matrix of the plate and $\{c\}$, $\{d\}$ and $\{e\}$ are the column matrices of the unknown constants, respectively, given by

$$\{c\} = [c_{11}, c_{12}, \ldots, c_{1J}, c_{21}, c_{22}, \ldots, c_{2J}, \ldots, c_{I1}, c_{I2}, \ldots, c_{IJ}]^{\rm T},$$

$$\{d\} = [d_{11}, d_{12}, \ldots, d_{1J}, d_{21}, d_{22}, \ldots, d_{2J}, \ldots, d_{I1}, d_{I2}, \ldots, d_{IJ}]^{\rm T}, \tag{6}$$

$$\{e\} = [e_{11}, e_{12}, \ldots, e_{1J}, e_{21}, e_{22}, \ldots, e_{2J}, \ldots, e_{I1}, e_{I2}, \ldots, e_{IJ}]^{\rm T}$$

and

$$[K] = \begin{bmatrix} [K_{cc}] & [K_{cd}] & [K_{ce}] \\ & [K_{dd}] & [K_{de}] \\ \text{symmetric} & & [K_{ee}] \end{bmatrix}, \qquad [M] = \begin{bmatrix} [M_{cc}] & [M_{cd}] & [M_{ce}] \\ & [M_{dd}] & [M_{de}] \\ \text{symmetric} & & [M_{ee}] \end{bmatrix}, \tag{7}$$

where

$$
\begin{aligned}
K_{ccimjn} &= (E_{im}^{1,1}F_{jn}^{0,0} + \lambda^2 E_{im}^{0,0}F_{jn}^{1,1})/R + [\alpha_{x0}X_i(0)X_m(0) + \alpha_{x1}X_i(1)X_m(1)]F_{jn}^{0,0} \\
&\quad + \lambda^4 E_{im}^{0,0}[\alpha_{y0}Y_j(0)Y_n(0) + \alpha_{y1}Y_j(1)Y_n(1)], \\
K_{cdimjn} &= T_{im}^{1,0}F_{jn}^{0,0}/R, \qquad K_{ceimjn} = \lambda^2 E_{im}^{0,0}U_{jn}^{1,0}/R, \\
K_{ddimjn} &= G_{im}^{1,1}F_{jn}^{0,0} + (1-\nu)\lambda^2 G_{im}^{0,0}F_{jn}^{1,1}/2 + G_{im}^{0,0}F_{jn}^{0,0}/R + [\beta_{x0}\Phi_i(0)\Phi_m(0) + \beta_{x1}\Phi_i(1)\Phi_m(1)]F_{jn}^{0,0}, \\
K_{deimjn} &= \lambda^2[\nu A_{im}^{1,0}U_{jn}^{0,1} + (1-\nu)A_{im}^{0,1}U_{jn}^{1,0}/2], \\
K_{eeimjn} &= \lambda^4 E_{im}^{0,0}H_{jn}^{1,1} + (1-\nu)\lambda^2 E_{im}^{1,1}H_{jn}^{0,0}/2 + \lambda^2 E_{im}^{0,0}H_{jn}^{0,0}/R \\
&\quad + \lambda^4 E_{im}^{0,0}[\beta_{y0}\Psi_i(0)\Psi_m(0) + \beta_{y1}\Psi_i(1)\Psi_m(1)], \\
\Omega^2 &= \rho h b^4\omega^2/(D\pi^4), \\
M_{ccimjn} &= E_{im}^{0,0}F_{jn}^{0,0}, \qquad M_{ddimjn} = \gamma^2 G_{im}^{0,0}F_{jn}^{0,0}/(12\lambda^2), \\
M_{eeimjn} &= \gamma^2 E_{im}^{0,0}H_{jn}^{0,0}/12, \qquad M_{cdimjn} = M_{ceimjn} = M_{deimjn} = 0
\end{aligned} \tag{8}
$$

in which,

$$
\begin{aligned}
E_{im}^{r,s} &= \int_0^1 \frac{\mathrm{d}^r X_i(\xi)}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s X_m(\xi)}{\mathrm{d}\xi^s}\mathrm{d}\xi, \quad G_{im}^{r,s} = \int_0^1 \frac{\mathrm{d}^r \Phi_i(\xi)}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s \Phi_m(\xi)}{\mathrm{d}\xi^s}\mathrm{d}\xi, \\
T_{im}^{r,s} &= \int_0^1 \frac{\mathrm{d}^r X_i(\xi)}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s \Phi_m(\xi)}{\mathrm{d}\xi^s}\mathrm{d}\xi, \quad A_{im}^{r,s} = \int_0^1 \frac{\mathrm{d}^r \Phi_i(\xi)}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s X_m(\xi)}{\mathrm{d}\xi^s}\mathrm{d}\xi, \\
F_{jn}^{r,s} &= \int_0^1 \frac{\mathrm{d}^r Y_j(\eta)}{\mathrm{d}\eta^r}\frac{\mathrm{d}^s Y_n(\eta)}{\mathrm{d}\eta^s}\mathrm{d}\eta, \quad H_{jn}^{r,s} = \int_0^1 \frac{\mathrm{d}^r \Psi_j(\eta)}{\mathrm{d}\eta^r}\frac{\mathrm{d}^s \Psi_n(\eta)}{\mathrm{d}\eta^s}\mathrm{d}\eta, \\
U_{jn}^{r,s} &= \int_0^1 \frac{\mathrm{d}^r Y_j(\eta)}{\mathrm{d}\eta^r}\frac{\mathrm{d}^s \Psi_n(\eta)}{\mathrm{d}\eta^s}\mathrm{d}\eta, \qquad (i,m = 1,2,3,\ldots,I;\ j,n = 1,2,3,\ldots,J;\ r,s = 0,1), \\
\alpha_{x0} &= k_{x0}^{\mathrm{t}}a^3/D, \quad \alpha_{x1} = k_{x1}^{\mathrm{t}}a^3/D, \quad \alpha_{y0} = k_{y0}^{\mathrm{t}}b^3/D, \quad \alpha_{y1} = k_{y1}^{\mathrm{t}}b^3/D, \\
\beta_{x0} &= k_{x0}^{\mathrm{r}}a/D, \quad \beta_{x1} = k_{x1}^{\mathrm{r}}a/D, \quad \beta_{y0} = k_{y0}^{\mathrm{r}}b/D, \quad \beta_{y1} = k_{y1}^{\mathrm{r}}b/D, \\
\lambda &= a/b, \qquad \gamma = h/b, \qquad R = \frac{D}{\kappa G h a^2} = \frac{\gamma^2}{6(1-\nu)\kappa\lambda^2}.
\end{aligned} \tag{9}
$$

Here Ω is referred to as the non-dimensional natural frequency. λ is referred to as the aspect ratio of the plate and γ is referred to as the thickness-span ratio of the plate. R is referred to as the shear correction coefficient of Mindlin plates.

3. A set of static Timoshenko beam functions

Investigate the static solutions of a uniform Timoshenko beam under a series of Fourier sinusoidal loads, which is a unit width strip taken from the particular Mindlin rectangular plate under consideration in a direction parallel to the edge of the plate. Without losing generality, here only a strip in the x direction is considered, as shown in Fig. 2. The Timoshenko beam theory is utilized to describe the stress-placement relations as follows

$$
M(x) = -EI\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}, \qquad V(x) = -GA\kappa_x\left(\psi(x) - \frac{\mathrm{d}z(x)}{\mathrm{d}x}\right), \tag{10}
$$

where $M(x)$ is the bending moment of the beam, $V(x)$ is the transverse shear force of the beam, $\psi(x)$ is the rotation of the cross-section of the beam, $z(x)$ is the transverse deflection of centroidal axis of the beam, κ_x is the shear correction factor of the Timoshenko beam, EI is the flexural rigidity of the beam, G is the shear

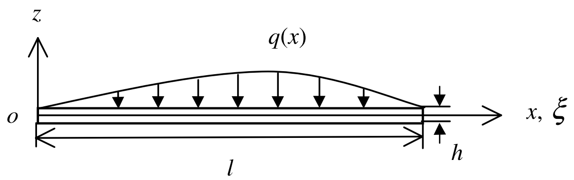

Fig. 2. The uniform Timoshenko beam under arbitrary load $q(x)$.

modulus of rigidity and A is the area of the cross-section of the beam. The equilibrium equations of stress are given by

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = V(x), \qquad \frac{\mathrm{d}V(x)}{\mathrm{d}x} = -q(x). \tag{11}$$

From Eqs. (10) and (11), one has

$$GA\kappa_x\left(\psi(x) - \frac{\mathrm{d}z(x)}{\mathrm{d}x}\right) = EI\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}, \qquad EI\frac{\mathrm{d}^3\psi(x)}{\mathrm{d}x^3} = q(x). \tag{12}$$

The load $q(x)$ in any case can be expanded into a Fourier sinusoidal series in the interval $(0, a)$ as follows

$$q(x) = (EI/a^4)\sum_{i=1}^{\infty} Q_i(\mathrm{i}\pi)^4 \sin(\mathrm{i}\pi x/a). \tag{13}$$

Introducing the following non-dimensional coordinate and parameter

$$\xi = x/a, \qquad R_x = \frac{EI}{GA\kappa_x a^2}, \tag{14}$$

the solutions of Eq. (12) can be written as

$$z = \sum_{i=1}^{\infty} Q_i z_i(\xi), \qquad \psi = \sum_{i=1}^{\infty} Q_i \psi_i(\xi)/a, \tag{15}$$

in which,

$$z_i(\xi) = C_0^i + C_1^i\xi + C_2^i\xi^2/2 + C_3^i(\xi^3/6 - R_x\xi) + [R_x(\mathrm{i}\pi)^2 + 1]\sin(\mathrm{i}\pi\xi), \tag{16}$$

$$\psi_i(\xi) = C_1^i + C_2^i\xi + C_3^i\xi^2/2 + \mathrm{i}\pi\cos(\mathrm{i}\pi\xi), \tag{17}$$

in the above two equations, C_j^i $(j = 0, 1, 2, 3)$ are the unknown constants, which can be evaluated by the boundary conditions of the beam, given by the following equations

$$M(0) = -k_{x0}^{\mathrm{r}}\psi(0), \qquad M(1) = k_{x1}^{\mathrm{r}}\psi(1), \qquad V(0) = k_{x0}^{\mathrm{t}}z(0), \qquad V(1) = -k_{x1}^{\mathrm{t}}z(1). \tag{18}$$

Substituting Eqs. (10), (16) and (17) into the above equations, one has

$$\begin{aligned} &C_0^i = (a_{22}f_1^i - a_{12}f_2^i)/\Delta, \qquad C_1^i = (a_{11}f_2^i - a_{21}f_1^i)/\Delta, \qquad C_3^i = -\alpha_{x0}C_0^i + (\mathrm{i}\pi)^3, \\ &C_2^i = \beta_{x0}(C_1^i + \mathrm{i}\pi), \end{aligned} \tag{19}$$

where

$$\begin{aligned} &a_{11} = \alpha_{x0} + \alpha_{x1} - \alpha_{x0}\alpha_{x1}(1/6 - R_x), \qquad a_{12} = \alpha_{x1}(1 + \beta_{x0}/2), \\ &a_{21} = \alpha_{x0}(1 + \beta_{x1}/2), \qquad a_{22} = -(\beta_{x0} + \beta_{x1} + \beta_{x0}\beta_{x1}), \qquad \Delta = a_{11}a_{22} - a_{12}a_{21}, \\ &f_1^i = (i\pi)^3[1 - (-1)^i - \alpha_{x1}(1/6 - R_x)] - \alpha_{x1}\beta_{x0}(i\pi)/2, \\ &f_2^i = i\pi[\beta_{x0} + (-1)^i\beta_{x1} + \beta_{x0}\beta_{x1}] + (i\pi)^3(1 + \beta_{x1}/2). \end{aligned} \tag{20}$$

It can be easily demonstrated that the solutions of C_j^i $(j = 0, 1, 2, 3)$ in Eq. (19) always exist because of

$$\Delta = \alpha_{x0}\alpha_{x1}\beta_{x0}(1/3 + R_x) + \alpha_{x1}\beta_{x0}\beta_{x1}(1/3 + R_x) + \alpha_{x0}\alpha_{x1}\beta_{x0}\beta_{x1}(1/12 + R_x) > 0, \tag{21}$$

except that the boundary constraints of the beam cannot remove the rigid body movements from the beam. However, for the beams permitting rigid body movements one can readily add the modes of the rigid body movements into the solutions described by Eq. (15). For example, if the beam is free–free, one can take the first and the second beam functions, respectively, as

$$z_1 = 1, \qquad \psi_1 = 0 \quad \text{and} \quad z_2 = \xi - 0.5, \qquad \psi_2 = 1, \tag{22}$$

the third and higher beam functions supplied by the set of simply–simply supported Timoshenko beam functions which can be obtained by letting $\alpha_{x0} = \alpha_{x1} = \infty$ and $\beta_{x0} = \beta_{x1} = 0$ in the above analysis. If the beam is restrained only by a translational spring on one edge and the other one is free, then one can take the first beam function as

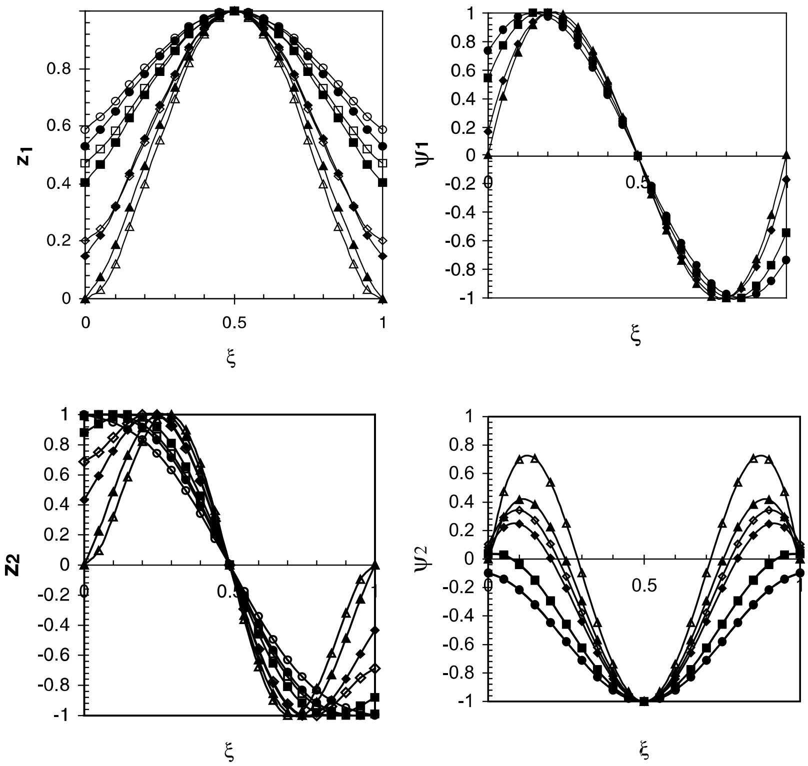

Fig. 3. The first two static Timoshenko beam functions for the beam with symmetrically elastically restrained edges: (○) denotes $\alpha = 50$, $\beta = 5$, $\gamma = 0.001$ and (●) denotes $\alpha = 50$, $\beta = 5$, $\gamma = 0.2$. (□) denotes $\alpha = 100$, $\beta = 10$, $\gamma = 0.001$ and (■) denotes $\alpha = 100$, $\beta = 10$, $\gamma = 0.2$; (◇) denotes $\alpha = 500$, $\beta = 50$, $\gamma = 0.001$ and (◆) denotes $\alpha = 500$, $\beta = 50$, $\gamma = 0.2$; (△) denotes $\alpha = 10^8$, $\beta = 10^8$, $\gamma = 0.001$ and (▲) denotes $\alpha = 10^8$, $\beta = 10^8$, $\gamma = 0.2$.

Table 1
Convergency study of frequency parameter $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for a thin square plate ($h/b = 0.001$) with four edges equally elastically restrained

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$\alpha = 8$, $\beta = 2$									
1 × 1	0.5583								
2 × 2	0.5570	0.9798	0.9798	1.8074					
3 × 3	0.5541	0.9647	0.9647	1.7972	2.8893	3.0500	4.0610	4.0610	6.8583
4 × 4	0.5541	0.9643	0.9643	1.7941	2.8891	3.0497	4.0513	4.0513	6.8572
5 × 5	0.5538	0.9627	0.9627	1.7934	2.8352	3.0249	4.0391	4.0391	6.8520
6 × 6	0.5538	0.9627	0.9627	1.7923	2.8352	3.0248	4.0354	4.0354	6.8518
7 × 7	0.5537	0.9620	0.9620	1.7921	2.8240	3.0224	4.0344	4.0344	6.8412
8 × 8	0.5537	0.9619	0.9619	1.7918	2.8240	3.0224	4.0325	4.0325	6.8411
Xiang et al. (1997)	0.5537	0.9605	0.9605	1.7720	2.7898	3.0169	3.9790	3.9790	6.7436
$\alpha = 100$, $\beta = 100$									
1 × 1	1.7890								
2 × 2	1.7854	2.6215	2.6215	3.4891					
3 × 3	1.7605	2.5832	2.5832	3.4881	4.7096	4.7710	5.8178	5.8178	8.5193
4 × 4	1.7595	2.5649	2.5649	3.4382	4.7092	4.7705	5.7775	5.7775	8.5179
5 × 5	1.7576	2.5626	2.5626	3.4382	4.6595	4.7228	5.7355	5.7355	8.4348
6 × 6	1.7575	2.5612	2.5612	3.4342	4.6593	4.7226	5.7320	5.7320	8.4346
7 × 7	1.7571	2.5608	2.5608	3.4342	4.6535	4.7171	5.7274	5.7274	8.4256
8 × 8	1.7571	2.5605	2.5605	3.4335	4.6534	4.7170	5.7266	5.7266	8.4255
Xiang et al. (1997)	1.7566	2.5599	2.5599	3.4317	4.6483	4.7135	5.7184	5.7184	8.4111
$\alpha = 1176$, $\beta = 98$									
1 × 1	3.1660								
2 × 2	3.1660	5.8184	5.8184	7.9992					
3 × 3	3.1252	5.7540	5.7540	7.9948	8.7242	8.9858	10.838	10.838	13.417
4 × 4	3.1245	5.6948	5.6848	7.7064	8.7240	8.9855	10.577	10.577	12.843
5 × 5	3.1080	5.6715	5.6715	7.6982	8.5471	8.8844	10.406	10.406	12.778
6 × 6	3.1075	5.6489	5.6489	7.6142	8.5467	8.8840	10.336	10.336	12.542
7 × 7	3.1032	5.6422	5.6422	7.6108	8.5057	8.8603	10.296	10.296	12.527
8 × 8	3.1030	5.6352	5.6352	7.5870	8.5055	8.8601	10.277	10.277	12.477
Xiang et al. (1997)	3.1006	5.6256	5.6256	7.5669	8.4857	8.8481	10.244	10.244	12.447

$$z_1 = \xi, \qquad \psi_1 = 1 \tag{23}$$

for the beam with a free edge at $\xi = 1$ and

$$z_1 = \xi - 1, \qquad \psi_1 = 1 \tag{24}$$

for the beam with the free edge at $\xi = 0$. The second and higher beam functions supplied by such a set of beam functions in which the free edge of the beam has been changed into the simply-supported edge. If the beam is restrained only by rotational springs on edges, one can take the first beam function as fol-lows

$$z_1 = 1, \qquad \psi_1 = 0, \tag{25}$$

the second and higher beam functions supplied by such a set of beam functions in which a zero transverse displacement has been added to either end of the beam. It is clear that in the above analysis, letting the elastic stiffness of the springs approach zero or infinity can give all kinds of classical boundary conditions of the beam. In actual numerical computations, in order to maintain the generality of the program, infinite stiffness can be represented through giving large quantities such as 10^8.

The static Timoshenko beam functions in the y direction can also be written down only by changing the parameters in the x direction into those in the y direction, such as a is replaced by b, R_x is replaced by R_y, α_{x0} and α_{x1} are replaced by α_{y0} and α_{y1} respectively, β_{x0} and β_{x1} are replaced by β_{y0} and β_{y1} respectively. In

Table 2
Convergency study of frequency parameter $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for a moderately thick square plate ($h/b = 0.2$) with four edges equally elastically restrained

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$\alpha = 10,\ \beta = 5$									
1×1	0.6214								
2×2	0.6186	1.1024	1.1024	1.8098					
3×3	0.6156	1.0866	1.0866	1.7949	2.8518	2.9045	3.5693	3.5693	5.2330
4×4	0.6155	1.0864	1.0864	1.7931	2.8516	2.9042	3.5656	3.5656	5.2322
5×5	0.6152	1.0837	1.0837	1.7910	2.7981	2.8605	3.5330	3.5330	5.1992
6×6	0.6152	1.0837	1.0837	1.7892	2.7979	2.8603	3.5299	3.5299	5.1988
7×7	0.6151	1.0826	1.0826	1.7885	2.7861	2.8522	3.5238	3.5238	5.1908
8×8	0.6151	1.0826	1.0826	1.7872	2.7860	2.8521	3.5215	3.5215	5.1906
$\alpha = 50,\ \beta = 10$									
1×1	1.2448								
2×2	1.2423	1.9082	1.9082	2.5911					
3×3	1.2281	1.8832	1.8832	2.5904	3.4684	3.5241	4.1891	4.1891	5.7338
4×4	1.2276	1.8729	1.8729	2.5655	3.4681	3.5239	4.1746	4.1746	5.7336
5×5	1.2258	1.8693	1.8693	2.5651	3.4042	3.4657	4.1282	4.1282	5.6655
6×6	1.2256	1.8682	1.8682	2.5620	3.4041	3.4656	4.1256	4.1256	5.6653
7×7	1.2251	1.8669	1.8669	2.5618	3.3903	3.4537	4.1171	4.1171	5.6538
8×8	1.2250	1.8667	1.8667	2.5608	3.3901	3.4536	4.1158	4.1158	5.6536
$\alpha = 100,\ \beta = 100$									
1×1	1.6538								
2×2	1.6516	2.4411	2.4411	3.2025					
3×3	1.6333	2.4119	2.4119	3.2020	4.0087	4.0702	4.7899	4.7899	6.3036
4×4	1.6323	2.3927	2.3927	3.1506	4.0084	4.0699	4.7582	4.7582	6.3036
5×5	1.6298	2.3891	2.3891	3.1506	3.9494	4.0148	4.7095	4.7095	6.2152
6×6	1.6295	2.3863	2.3863	3.1428	3.9493	4.0147	4.7045	4.7045	6.2151
7×7	1.6287	2.3853	2.3853	3.1428	3.9375	4.0038	4.6952	4.6952	6.1982
8×8	1.6286	2.3846	2.3846	3.1405	3.9374	4.0038	4.6936	4.6936	6.1981
Xiang et al. (1997)	1.6216	2.3760	2.3760	3.1281	3.9096	3.9773	4.6584	4.6584	6.1357

keeping with the fact that the beam under consideration is a unit width strip taken from the plate, the following relations can be utilized

$$A = h, \qquad EI = (1 - \nu^2)D, \qquad \kappa_x = \kappa, \qquad R_x = (1 - \nu^2)R. \tag{26}$$

For the small Poisson's ratio ν (in most cases, $\nu = 0.3$ or $1/3$), the flexural stiffness EI of the beam also can be replaced by the flexural stiffness D of the plate and the shear correction coefficient of the beam R_x can be replaced by the shear correction coefficient of the Mindlin plate R without significant errors. The first two static Timoshenko beam functions are given in Fig. 3, with respect to different support stiffness and thickness-span ratio of the beam. It can be seen that this set of static Timoshenko beam functions varies with the support stiffness and on which the effect of shear correction factor is obvious, especially for the beam with large rotational stiffness.

4. Comparison and convergency studies

In the following computations, The Gaussian integral formulation of 24 points is used numerically to evaluate the integrals in Eq. (9). Poisson's ratio $\nu = 0.3$ and the shear correction factor $\kappa = 5/6$ are used in

Table 3
Comparison study of frequency parameter $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for a moderately thick square plate ($h/b = 0.1$) with elastically restrained edges

Sources	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
The plate with three clamped edges and the remaining one elastically restrained: $\alpha = \beta = 100$									
Present	2.6820	4.3394	5.7503	7.2321	7.6054	9.9868	10.139	11.289	12.054
Saha et al. (1996)	2.6849	4.3510	5.7553	7.2551					
The plate with four edges elastically restrained: $\alpha = \beta = 10$									
Present	0.6238	1.1635	1.1635	1.9763	3.3646	3.4020	4.2661	4.2661	6.6049
Saha et al. (1996)	0.6237	1.1631	1.1631	1.9769					
The plate with four edges elastically restrained: $\alpha = \beta = 1000$									
Present	2.9176	5.0656	5.0656	6.7119	7.4491	7.7571	8.9463	8.9463	10.732
Saha et al. (1996)	2.9227	5.0630	5.0630	6.7002					
The simply supported plate with four edges elastically restrained against rotation: $\beta = 10$									
Present	2.7053	5.4355	5.4355	7.8676	9.3848	9.4115	11.535	11.535	14.213
Chung et al. (1993)	2.704	5.438	5.440	7.872	9.398	9.425			
The simply supported plate with four edges elastically restrained against rotation: $\beta = 50$									
Present	3.1126	5.9980	5.9980	8.4774	10.013	10.083	12.172	12.172	14.861
Chung et al. (1993)	3.111	5.999	6.001	8.482	10.021	10.094			
The simply supported plate with four edges elastically restrained against rotation: $\beta = 100$									
Present	3.1970	6.1266	6.1266	8.6246	10.170	10.251	12.336	12.336	15.035
Chung et al. (1993)	3.194	6.123	6.130	8.630	10.175	10.260			
The simply supported plate with four edges elastically restrained against rotation: $\beta = 1000$									
Present	3.2845	6.2643	6.2643	8.7856	10.344	10.437	12.520	12.520	15.233
Chung et al. (1993)	3.281	6.263	6.267	8.792	10.345	10.443			
The plate with four edges only elastically restrained against translation: $\alpha = 3.4404$, $\beta = 0$									
Present	0.3673	0.5260	0.5260	1.4307	2.0086	2.4610	3.3405	3.3405	5.6911
Gorman (1997b)	0.3673	0.5260	0.5260	1.404					
The plate with four edges only elastically restrained against translation: $\alpha = 34.404$, $\beta = 0$									
Present	0.9748	1.5859	1.5859	2.2944	2.6803	3.0556	3.8681	3.8681	5.9599
Gorman (1997b)	0.9720	1.582	1.582	2.285					

the computations. The static Timoshenko beam functions derived in the last section are used as the admissible functions of the Mindlin rectangular plates. Taking the admissible functions of the plates in the x direction as an example, one has

$$X_i(\xi) = z_i(\xi), \qquad \Phi_i(\xi) = \psi_i(\xi). \tag{27}$$

Similarly, one can also obtain the admissible functions of the plates in the y direction. It should be pointed out that using Timoshenko beam functions as the admissible functions of Mindlin rectangular plates are sound theoretically because of the fact that the Timoshenko beam is a one-dimensional equivalent of the Mindlin rectangular plates such as the Bernoulli–Euler beam is a one-dimensional equivalent of the classical thin plates.

Table 4
The first nine frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for moderately thick rectangular plates ($h/b = 0.2$) with two opposite free edges, the other two edges only elastically restrained against translation ($\beta = 0$)

α^t_{x0}, α^t_{x1}	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$a/b = 1$									
5, 5	0.30522	0.30755	0.53964	1.3117	1.8447	2.2089	2.8913	2.9082	4.6247
50, 50	0.69412	0.82250	1.5797	1.9268	2.1634	2.7919	3.1839	3.3823	4.7189
500, 500	0.88307	1.3081	2.7583	2.8767	3.2089	4.3584	5.0980	5.1368	5.3785
5000, 5000	0.91051	1.4254	2.9541	3.1485	3.6157	4.9856	5.3230	5.9648	6.3290
5, 10	0.34010	0.36813	0.66584	1.3584	1.8740	2.2377	2.9090	2.9365	4.6335
10, 50	0.47942	0.60383	1.1781	1.6751	2.0587	2.5540	3.0600	3.1878	4.6769
100, 500	0.82757	1.1387	2.3416	2.5060	2.6785	3.8583	4.1292	4.5195	4.9306
1000, 5000	0.90423	1.3980	2.9093	3.0851	3.5166	4.8267	5.2811	5.7547	6.0876
$a/b = 1.5$									
5, 5	0.13599	0.13816	0.24271	0.84976	0.95270	1.8234	1.9983	2.2477	2.5494
50, 50	0.31351	0.39951	0.71804	1.0540	1.2802	1.9696	2.0423	2.3675	2.6378
500, 500	0.40273	0.75582	1.3796	1.7131	2.2817	2.5097	2.8339	2.9816	3.6500
5000, 5000	0.41617	0.86850	1.5420	2.0496	2.4530	3.1091	3.5199	3.5600	4.8988
5, 10	0.15171	0.16734	0.29969	0.86335	0.97342	1.8318	2.0011	2.2543	2.5539
10, 50	0.21461	0.29845	0.53660	0.96975	1.1618	1.9067	2.0226	2.3167	2.5980
100, 500	0.37629	0.61880	1.0915	1.4356	1.9688	2.1702	2.4636	2.7804	3.2289
1000, 5000	0.41304	0.84021	1.5029	1.9582	2.4079	2.9520	3.3495	3.3585	4.5211
$a/b = 0.5$									
5, 5	1.1945	1.2042	2.0331	2.2834	2.9279	4.6904	4.7271	6.6312	6.9051
50, 50	2.5647	2.7103	3.6292	5.6359	5.6625	5.9547	6.8641	8.3092	8.3934
500, 500	3.1038	3.5380	4.8610	6.8614	8.7036	8.9583	9.3831	9.8643	11.264
5000, 5000	3.1714	3.6498	5.0368	7.0516	9.1189	9.4124	9.5565	10.411	11.890
5, 10	1.3345	1.3801	2.4003	2.5135	3.2390	4.7699	4.9048	6.8016	7.0137
10, 50	1.8471	1.9933	2.9830	4.2542	4.6560	5.2382	5.9621	7.7655	7.9294
100, 500	2.9562	3.2926	4.4604	6.4254	7.7099	7.9454	8.7612	9.0226	10.192
1000, 5000	3.1562	3.6257	4.9996	7.0113	9.0269	9.3124	9.5195	10.291	11.752

In order to evaluate the accuracy of the present method, first of all, comparison and convergency studies are carried out in this section. Tables 1 and 2, respectively, give the first nine frequency parameters for thin ($h/b = 0.001$) and moderately thick ($h/b = 0.2$) square plates with four edges equally elastically restrained ($\alpha^t_{x0} = \alpha^t_{x1} = \alpha^t_{y0} = \alpha^t_{y0} = \alpha$, $\beta^r_{x0} = \beta^r_{x1} = \beta^r_{y0} = \beta^r_{y1} = \beta$), with respect to the number of terms of the static Timoshenko beam functions used. From these two tables, the rapid and monotonically downward convergency is observed for both thin plate ($h/b = 0.001$) and moderately thick plate ($h/b = 0.2$). The comparison of results with those available from Xiang et al. (1997) shows that rather satisfactory accuracy can be obtained by using only a small number of terms of the static Timoshenko beam functions. It can be seen that the maximum error of the fundamental frequency is less than 2.1% even if only one term of the static Timoshenko beam function is used in the computations. A further comparison study, given in Table 3, is made for moderately thick rectangular plates ($h/b = 0.1$) with various boundary conditions. Eight terms of the static beam functions in each direction are used. In order to provide reasonable comparison, the values of both Poisson's ratio and the shear correction factor are the same as those used in literature such as $\nu = 0.3$ and $\kappa = 0.85$ used by Saha et al. (1996), $\nu = 0.3$ and $\kappa = \pi^2/12$ used by Chung et al. (1993), $\nu = 0.333$ and $\kappa = 0.8601$ used by Gorman (1997b). Good agreement is observed for all cases. It should be noted that by letting both R_x and R_y equal to zero, this set of static Timoshenko beam functions will automatically degenerate into a set of static Bernoulli–Euler beam functions which has been successfully applied to the vibration analysis of thin rectangular plates with edges elastically restrained

Table 5
The first nine frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for moderately thick rectangular plates ($h/b = 0.2$) with two opposite free edges, the other two edges simply supported and elastically restrained against rotation ($\alpha = 0$)

β^r_{x0}, β^r_{x1}	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$a/b = 1$									
1, 1	1.0579	1.5158	2.9905	3.2858	3.7434	5.0928	5.3404	6.1301	6.5002
5, 5	1.3396	1.6953	3.0489	3.5375	3.9422	5.1929	5.3569	6.3116	6.6577
50, 50	1.6977	1.9578	3.1518	3.9415	4.2723	5.3722	5.3892	6.6361	6.9389
500, 500	1.7724	2.0169	3.1783	4.0378	4.3529	5.3983	5.4188	6.7193	7.0112
1, 5	1.1981	1.6040	3.0193	3.4143	3.8438	5.1428	5.3486	6.2219	6.5797
2, 10	1.3180	1.6837	3.0468	3.5317	3.9972	5.1907	5.3566	6.3097	6.6556
10, 50	1.5876	1.8750	3.1182	3.8155	4.1680	5.3143	5.3784	6.5330	6.8493
100, 500	1.7548	2.0030	3.1720	4.0150	4.3338	5.3961	5.4076	6.6995	6.9904
$a/b = 1.5$									
1, 1	0.48893	0.90750	1.6238	2.1316	2.4700	3.2288	3.5988	3.6991	4.9598
5, 5	0.63406	0.98241	1.7831	2.2394	2.4875	3.3670	3.6401	3.8065	4.9642
50, 50	0.83479	1.1167	2.0791	2.4591	2.5284	3.6739	3.7380	4.0513	4.9762
500, 500	0.87959	1.1520	2.1587	2.5226	2.5419	3.7681	3.7706	4.1288	4.9807
1, 5	0.56069	0.94416	1.7045	2.1855	2.4787	3.2986	3.6194	3.7531	4.9620
2, 10	0.62341	0.97937	1.7817	2.2391	2.4877	3.3700	3.6409	3.8087	4.9644
10, 50	0.77097	1.0721	1.9822	2.3855	2.5141	3.5708	3.7042	3.9681	4.9719
100, 500	0.86895	1.1435	2.1395	2.5072	2.5385	3.7452	3.7626	4.1098	4.9796
$a/b = 0.5$									
1, 1	3.5581	3.9573	5.1983	7.1250	9.3337	9.5935	9.6176	10.580	12.022
5, 5	4.1942	4.4725	5.4667	7.2328	9.6226	9.6360	9.8851	10.773	12.139
50, 50	4.8208	4.9958	5.7586	7.3567	9.6868	9.8844	10.130	10.956	12.256
500, 500	4.9293	5.0878	5.8119	7.3802	9.6967	9.9226	10.167	10.985	12.275
1, 5	3.8754	4.2128	5.3309	7.1785	9.4845	9.6153	9.7567	10.679	12.079
2, 10	4.1319	4.4224	5.4420	7.2239	9.5983	9.6361	9.8648	10.757	12.126
10, 50	4.6426	4.8461	5.6739	7.3204	9.6714	9.8160	10.065	10.906	12.222
100, 500	4.9043	5.0666	5.7996	7.3748	9.6944	9.9140	10.159	10.978	12.271

(Zhou, 1995). However, directly using the static Bernoulli–Euler beam functions as the admissible functions of the Mindlin rectangular plates will result in obvious errors, especially for plates with large support stiffness. For example, the fundamental frequency parameter for the moderately thick square plate ($h/b = 0.2$) with four edges equally elastically restrained ($\alpha = \beta = 100$) is $\Omega_1 = 1.6422$ using eight terms of the static Bernoulli–Euler beam functions in each direction, which is higher than that ($\Omega_1 = 1.6333$) using only three terms of the static Timoshenko beam functions in each direction and the error will further increase with the increase of the stiffnesses of supports (for fully clamped plates, the error will reach a maximum).

5. Numerical results

In this section, some numerical results are given, which can serve as a supplement to the database of vibration characteristics of moderately thick plates. In all the following computations, eight terms of the static Timoshenko beam functions are used in each direction. Moreover, Poisson's ratio ν and shear correction factor κ have been set as 0.3 and 5/6, respectively. Tables 4–6 give the first nine frequency parameters Ω_i $(i = 1, 2, 3, \ldots 9)$ of moderately thick rectangular plates ($h/b = 0.2$) with two opposite free edges in the y direction and the other two edges elastically restrained. Figs. 4–6 give the first six frequency

Table 6
The first nine frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$ for moderately thick rectangular plates ($h/b = 0.2$) with two opposite free edges, the other two edges symmetrically elastically restrained

α, β	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$a/b = 1$									
5, 1	0.30842	0.30915	0.67543	1.3515	1.9451	2.3556	2.9109	3.0539	4.6297
5, 10	0.31037	0.31482	0.92088	1.4505	2.0122	2.8989	2.9659	3.4544	4.6399
5, 50	0.31108	0.31623	0.99682	1.4899	2.0197	2.9904	3.1320	3.6279	4.6469
5, 5000	0.31131	0.31664	1.0206	1.5034	2.0206	2.9987	3.2114	3.6886	4.6484
50, 1	0.75186	0.84065	1.5915	1.9336	2.1743	2.9133	3.1956	3.4775	4.7188
50, 10	0.86095	0.88542	1.6198	1.9519	2.1886	3.2256	3.2681	3.7498	4.7206
50, 50	0.89439	0.90323	1.6305	1.9599	2.1912	3.2374	3.4255	3.8729	4.7229
50, 5000	0.90481	0.90935	1.6341	1.9627	2.1912	3.2411	3.4807	3.9169	4.7229
5000, 1	1.0533	1.5036	2.9731	3.2493	3.6921	5.0184	5.3252	6.0254	6.3802
5000, 10	1.4705	1.7725	3.0511	3.6244	3.9847	5.1519	5.3370	6.2702	6.5865
5000, 50	1.6803	1.9274	3.1066	3.8593	4.1743	5.2460	5.3486	6.4380	6.7277
5000, 5000	1.7607	1.9901	3.1318	3.9578	4.2553	5.2883	5.3548	6.5115	6.7897
$a/b = 1.5$									
5, 1	0.13774	0.13828	0.30400	0.85773	1.0772	1.8643	1.9984	2.3353	2.5912
5, 10	0.13853	0.14027	0.41699	0.88235	1.3968	1.9991	2.0001	2.4508	2.9044
5, 50	0.13881	0.14090	0.45265	0.89482	1.5286	2.0022	2.0720	2.4673	3.1113
5, 5000	0.13887	0.14108	0.46389	0.89959	1.5739	2.0028	2.1005	2.4712	3.1906
50, 1	0.34025	0.40244	0.72340	1.0560	1.3514	2.0001	2.0417	2.4340	2.6873
50, 10	0.39092	0.41172	0.73671	1.0625	1.5590	2.0404	2.1026	2.5183	2.9857
50, 50	0.40652	0.41660	0.74193	1.0660	1.6533	2.0406	2.1594	2.5307	3.1730
50, 5000	0.41138	0.41849	0.74368	1.0674	1.6868	2.0404	2.1819	2.5333	3.2450
5000, 1	0.48663	0.89622	1.6010	2.0855	2.4556	3.1503	3.5274	3.5893	4.9115
5000, 10	0.70479	1.0089	1.8441	2.2443	2.4704	3.3419	3.5663	3.7276	4.9405
5000, 50	0.82398	1.0889	2.0196	2.3703	2.4867	3.5034	3.6056	3.8478	4.9406
5000, 5000	0.87186	1.1252	2.0998	2.4314	2.4963	3.5840	3.6280	3.9095	4.9407
$a/b = 0.5$									
5, 1	1.2047	1.2197	2.2967	2.5401	3.2566	4.7230	4.8828	6.9895	7.2492
5, 10	1.2211	1.2421	2.3161	3.3634	3.8513	4.7677	5.2103	7.1877	7.7619
5, 50	1.2255	1.2476	2.3205	3.5915	4.0262	4.7779	5.3199	7.2578	7.7820
5, 5000	1.2268	1.2492	2.3208	3.6604	4.0798	4.7789	5.3547	7.2792	7.7842
50, 1	2.7515	2.8444	3.6564	5.6279	5.7118	5.9919	6.8918	8.3192	8.4234
50, 10	3.0976	3.1052	3.7203	5.6141	5.8090	6.0667	6.9532	8.3210	8.4874
50, 50	3.1874	3.2011	3.7450	5.6105	5.8397	6.0910	6.9745	8.3188	8.5072
50, 5000	3.2133	3.2330	3.7530	5.6085	5.8494	6.0987	6.9813	8.3163	8.5126
5000, 1	3.5475	3.9428	5.1779	7.1036	9.2805	9.5613	9.5743	10.516	11.951
5000, 10	4.4462	4.6731	5.5573	7.2509	9.6278	9.6650	9.9175	10.771	12.103
5000, 50	4.7953	4.9653	5.7198	7.3181	9.6537	9.7981	10.043	10.864	12.162
5000, 5000	4.9136	5.0652	5.7768	7.3423	9.6634	9.8375	10.081	10.894	12.182

parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with four edges equally elastically restrained with respect to thickness-span ratio γ from 0.001 to 0.2. Figs. 7–9 give the first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with two opposite edges simply supported in the y direction and the other two edges symmetrically elastically restrained with respect to thickness-span ratio γ from 0.001 to 0.2. From these tables, it can be observed that the eigenfrequency parameters decrease as the thickness-span ratio γ increases. This is mainly due to the effect of the shear deformation and the rotary inertia. This effect becomes more pronounced for plates with greater edge constraints and for plates vibrating in higher modes.

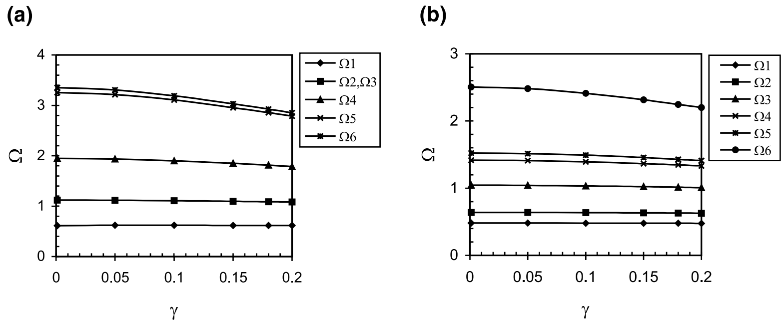

Fig. 4. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with four edges equally elastically restrained as the function of thickness-span ratio γ when $\alpha = 10$, $\beta = 5$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

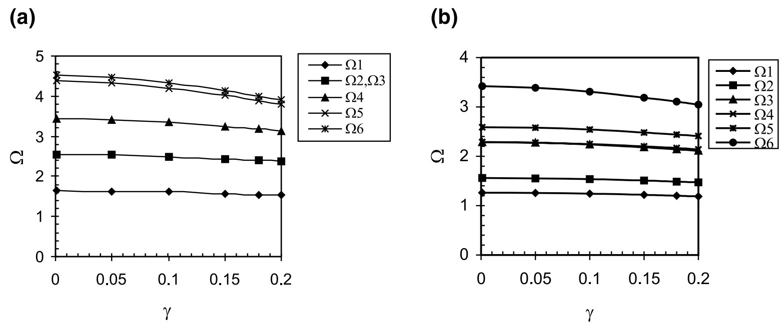

Fig. 5. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with four edges equally elastically restrained as the function of thickness-span ratio γ when $\alpha = 100$, $\beta = 10$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

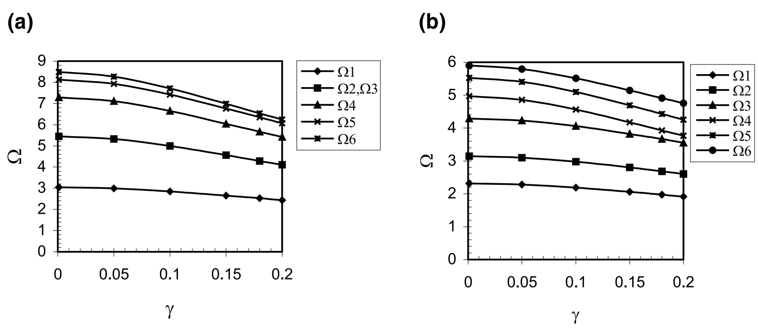

Fig. 6. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with four edges equally elastically restrained as the function of thickness-span ratio γ when $\alpha = 1000$, $\beta = 100$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

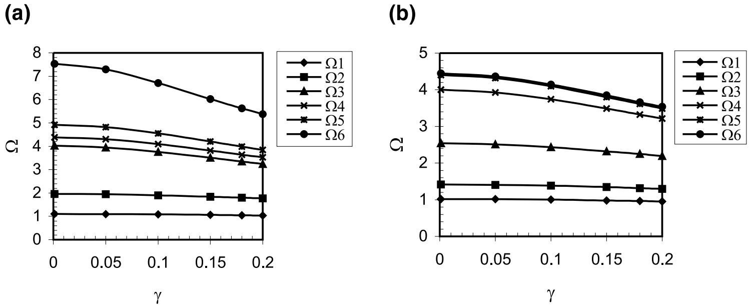

Fig. 7. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with two opposite edges simply supported and the other two edges symmetrically elastically restrained as the function of thickness-span ratio γ when $\alpha = 10$, $\beta = 5$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

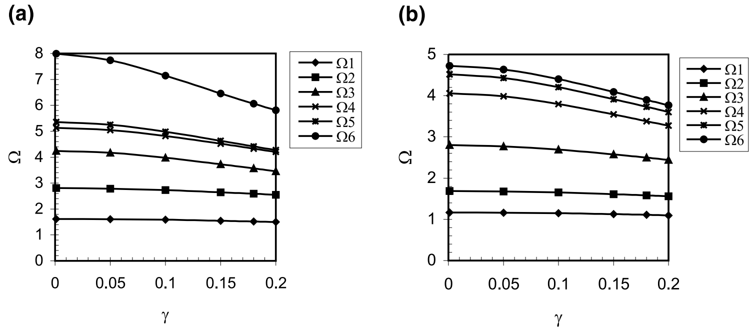

Fig. 8. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with two opposite edges simply supported and the other two edges symmetrically elastically restrained as the function of thickness-span ratio γ when $\alpha = 100$, $\beta = 10$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

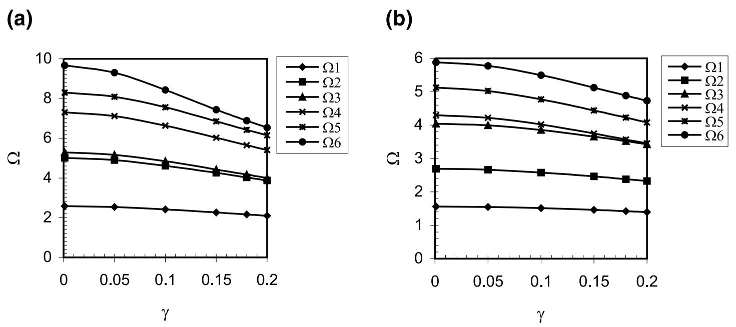

Fig. 9. The first six frequency parameters Ω_i $(i = 1, 2, 3, \ldots, 6)$ of Mindlin rectangular plates with two opposite edges simply supported and the other two edges symmetrically elastically restrained as the function of thickness-span ratio γ when $\alpha = 1000$, $\beta = 100$, (a) $\lambda = 1.0$ and (b) $\lambda = 1.5$.

6. Conclusions

In this paper, a set of static Timoshenko beam functions is developed as the admissible functions for the vibration analysis of moderately thick rectangular plates with elastically restrained edges in the Rayleigh–Ritz method. Each of the beam functions is a polynomial function of not more than third order plus a sine or cosine function. The coefficients of the polynomial are uniquely decided by the boundary conditions of the particular Mindlin rectangular plate under consideration. Thus a general program can be easily written for the plate with arbitrary thickness and support conditions. The convergency and comparison studies demonstrate the high accuracy and low computational cost of the present method. Some meaningful results are provided as a supplement to database of vibration characteristics of moderately thick plates.

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